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Disjunctive and conjunctive representations in finite lattices and convexity spaces

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Abstract

The concepts of disjunctive and conjunctive form, implicants and implicata, well known in lattices of Boolean functions, are examined in the general context of finite lattices and finite convexity spaces. The validity of the Blake–Quine consensus procedure for the determination of the prime implicants is shown to depend on a simple form of join reducibility. In the context of convexity spaces, another algebraic procedure for the determination of the prime implicants, based on distributivity, is seen to be contingent on the Helly property for convex sets.

Keywords: Boolean functions; Consensus; Resolution; Implicants; Prime implicants; Convexity; Helly property; DNF; CNF

1. Introduction

Consider a finite lattice L. Denote the join operation by $\vee$, and the meet by $\wedge$, or $\cdot$, or by juxtaposition.

Let us suppose that a subset G of L generates L as a join semilattice, i.e. that every element e of L is the join (least upper bound) of some $m \geqslant 0$ elements $c_1, \ldots, c_m$ of G,

$$e = c_1 \vee \cdots \vee c_m. \tag{1}$$

We shall say that G is a *join generating* set in L. The set $\{c_1, \ldots, c_m\}$, which determines the right-hand side of (1) up to the order and possible repetition of the c_i's, is said to constitute a *disjunctive representation* of e. (We can also refer to the expression (1),

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or to the right-hand side of (1), as a "*disjunctive representation*" of e.) Any $c \in G$ such that $c \leqslant e$, whether appearing among the c_i in (1) or not, is called an *implicant* of e. An implicant c that is maximal among the implicants of e is called a *prime implicant* of e. The set of all prime implicants of an element e constitutes a particular disjunctive representation of e (see [8]).

The above terminology originates in the theory of Boolean functions. The set L of *n-adic Boolean functions* ($n \geqslant 1$), i.e. functions $\{0,1\}^n \to \{0,1\}$, is a lattice. In this lattice $f \leqslant g$ if and only if

$$\forall \mathbf{v} \in \{0,1\}^n \quad f(\mathbf{v}) = 1 \;\Rightarrow\; g(\mathbf{v}) = 1.$$

Among the n-adic functions, we have the n *projection* functions (or "*variables*") $x_1, \ldots, x_n$ given by

$$x_i(v_1, \ldots, v_i, \ldots, v_n) = v_i$$

and their complements $\bar{x}_i, \ldots, \bar{x}_n$ in L. (In fact L is a Boolean lattice, so that every $x \in L$ has a unique complement $\bar{x}$.) The n variables and their n complements are called *literals*, and the meet (greatest lower bound) of any $m \geqslant 0$ literals

$$l_1 \wedge \cdots \wedge l_m = l_1 \ldots l_m,$$

where no l_i equals $\bar{l}_j$, is called an *elementary conjunction*. An elementary conjunction determines uniquely the set of literals that appear in it. The set G of all n-adic elementary conjunctions is a join generating set in the lattice L of all n-adic Boolean functions, and it is in this context that the term "implicant" has been traditionally used. The disjunctive representations of the elements of a lattice correspond then to what is traditionally called a "disjunctive normal form" (or DNF) of a Boolean function.

The following algorithmic procedure, called "*consensus method*", to obtain the DNF formed by all the prime implicants of a Boolean function f, starting from any DNF of f as input, was proposed by Blake [6] and Quine [20]:

Start with any DNF

$$f = c_1 \vee \cdots \vee c_m \tag{2}$$

and perform, in any order, repeatedly the following two operations, until none can be performed any longer.

1. *Adjunction of consensus*: if for a variable x some c_i and c_j can be written as

   $$c_i = x c_i' \quad c_j = \bar{x} c_j'$$

   so that c_i' and c_j' are elementary conjunctions, and if there is no literal l appearing in c_i' so that $\bar{l}$ appears in c_j', then replace (2) by

   $$f = c_1 \vee \cdots \vee c_m \vee c_i' c_j', \tag{3}$$

   unless for some c_k we have $c_i' c_j' \leqslant c_k$.

2. *Absorption*: if $c_i \leqslant c_j$ for some $i \neq j$, then delete c_i from the join expression in (2).

In Section 2 we shall re-formulate the Boolean consensus algorithm in the language of lattices, and give a sufficient lattice-theoretical condition on G in order that the consensus procedure yield all the prime implicants. As elementary conjunctions obviously satisfy this condition, a particularly simple proof of the validity of the Blake–Quine consensus procedure will result. Moreover, this sufficient condition will turn out to be also necessary in the case of distributive lattices.

In Section 3 we shall suppose that the lattice L is the power set lattice $\mathscr{P}(X)$ of some finite ground set X (and thus it is a distributive lattice), and that G is the set of convex sets of some abstract convexity space defined on X. This will allow the generalization of a different procedure for the determination of prime implicants and implicates, by applying the law of distributivity in $\mathscr{P}(X)$. This method has also been known for Boolean functions, and we shall extend its validity to subsets of certain convexity spaces.

2. Consensus in finite lattices

It is interesting to notice that the elementary conjunction $c_i'c_j'$ appearing in (3) obtained when performing the "adjunction of consensus" step of the consensus algorithm for Boolean functions described in Section 1, is the unique maximal elementary conjunction c such that

$$c \leqslant c_i \vee c_j \quad c \neq c_i \quad c \neq c_j,$$

and in fact, the unique prime implicant c of $c_i \vee c_j$ such that $c \nleqslant c_k$ for all c_k.

Given a finite lattice L, and a join generating set G in L, we call a sequence $F_1,\ldots,F_t$ of subsets of G a *consensus sequence* if for each $2 \leqslant i \leqslant t$ one of the following conditions holds:

(i) F_i is obtained from F_{i-1} by removing a single element c of F_{i-1} such that $c \leqslant c'$ for some other $c' \in F_{i-1}$ ("*absorption*"),

(ii) F_i is obtained from F_{i-1} by adding a prime implicant c of some join $c_i \vee c_j$ (where $c_i, c_j \in F_{i-1}$) such that $c \nleqslant c_k$ for all $c_k \in F_{i-1}$ ("*adjunction of consensus*").

It is easy to see that for all i, j

$$\vee(c\colon\ c \in F_i) = \vee(c\colon\ c \in F_j).$$

In other words, if $F_1 = \{c_1,\ldots,c_m\}$, then each F_i, $i = 1,\ldots,t$ constitutes a different disjunctive representation of the same lattice element

$$e = c_1 \vee \cdots \vee c_m.$$

We shall say that $F_1,\ldots,F_t$ is a *consensus sequence for the element* e. It is easy to verify that for a given finite lattice L, an element deleted at an absorption step will not be reintroduced later by consensus adjunction, and therefore the possible length

t of consensus sequences is bounded by a constant. A consensus sequence $F_1,\dots,F_t$ is called *complete* if there is no $F_{t+1}\subseteq G$ such that $F_1,\dots,F_t$, F_{t+1} would also be a consensus sequence.

For an element $c\in G$ let us say that c is *G-prime* if

$$c \leqslant c_1 \vee \cdots \vee c_m$$

for $m\geqslant 2$ elements c_i of G, implies $c\leqslant c_i$ for at least one of the c_i.

Proposition 1. *Let L be a lattice, and let G be a join generating set in L. If*

(i) *every element of G is either G-prime or it is the join of two lesser elements of G, then*

(ii) *every complete consensus sequence for any $e\in L$ terminates with the disjunctive representation of e consisting of all of its prime implicants.*

If L is distributive, and G is closed under the meet operation, then (i) *and* (ii) *are equivalent.*

Proof. Assume (i). Let $F_1,\dots,F_t$ be a complete consensus sequence for some $e\in L$. If suffices to prove that every implicant of e is less than or equal to some member of F_t. Suppose this were not so for some implicant c, and suppose that c is minimal with respect to this property. Clearly c is not G-prime. From (i) it follows that $c=c_1\vee c_2$, where $c_1<c$, $c_2<c$. By the minimality of c, there exist c_1', c_2' in F_t such that $c_1\leqslant c_1'$, $c_2\leqslant c_2'$. But then $c\leqslant c_1'\vee c_2'$ and if we take any prime implicant c'' of $c_1'\vee c_2'$ such that $c\leqslant c''$, we have a longer consensus sequence

$$F_1,\dots,F_t,F_t\cup\{c''\},$$

contradicting the completeness of $F_1,\dots,F_t$. This shows that (i) implies (ii).

Suppose now that L is distributive, G is closed under meet, and assume (ii). Take an element $c\in G$ that is not G-prime. Then there exist some $c_1,\dots,c_m$ in G such that

$$c \leqslant c_1 \vee \cdots \vee c_m$$

but $c\nleqslant c_i$ for any c_i. By distributivity

$$c = cc_1 \vee \cdots \vee cc_m$$

and each cc_i is in G. Let

$$F_1=\{cc_1,\dots,cc_m\}.$$

Let now $F_1,\dots,F_t$ be a complete consensus sequence beginning with F_1. In view of (ii), c being an element of G, it is its own (unique) prime implicant, and hence $F_t=\{c\}$. Let j be the first index such that $c\in F_j$. Clearly $j>1$ and $c\leqslant k_1\vee k_2$ for some k_1, $k_2\in F_{j-1}$. Obviously k_1 and k_2 are smaller than c, and $c=k_1\vee k_2$. □

Example 1. If L is the lattice of all n-adic Boolean functions described in Section 1, and G is the set of elementary conjunctions, then condition (i) of Proposition 1 is easily verified. This yields a surprisingly simple proof of the validity of the Blake–Quine consensus method for the determination of the prime implicants of a Boolean function given in disjunctive normal form.

Example 2. If L is the dually ordered lattice of all n-adic Boolean functions, i.e. in which the order $\leqslant$ is given by

$$f \leqslant g \quad \text{if and only if } f(\mathbf{v}) = 0 \Rightarrow g(\mathbf{v}) = 0 \tag{4}$$

and if we take for G the set of all *elementary disjunctions*, i.e. the meets of any $m \geqslant 0$ literals in this dual lattice, then a (prime) implicant is what is usually called a (*prime*) *implicatum*. Condition (i) is again satisfied, thus providing an immediate proof of the validity of the “resolution method” for the determination of all prime implicata of a Boolean function (see e.g. Chang [7]).

Example 3. Let X be a Cartesian product of finite sets, $X = X_1 \times \cdots \times X_n$ and let L be the inclusion-ordered power set lattice $\mathscr{P}(X)$. Let

$$G = \{Y_1 \times \cdots \times Y_n\colon\ Y_i \subseteq X_i,\ i = 1, \ldots, n\}.$$

The members of G are referred to as “generalized rectangles” by Pichat [19], and Kaufmann and Pichat [13]. In the latter, an algorithm due originally to Malgrange [16] is presented for finding the maximal generalized rectangles contained in any given subset of X. It is easy to verify that condition (i) of Proposition 1 holds in this context. This validates the Malgrange algorithm, which proceeds essentially by constructing a complete consensus sequence. Those prime implicants of

$$(Y_1 \times \cdots \times Y_n) \cup (Z_1 \times \cdots \times Z_n)$$

which are distinct from $Y_1 \times \cdots \times Y_n$ and $Z_1 \times \cdots \times Z_n$ are among the n sets

$$(Y_1 \cap Z_1) \times \cdots \times (Y_{i-1} \cap Z_{i-1}) \times (Y_i \cup Z_i) \times (Y_{i+1} \cap Z_{i+1}) \times \cdots \times (Y_n \cap Z_n). \tag{5}$$

The Malgrange algorithm is in fact a generalization of the Blake–Quine algorithm. This can be seen by considering (as in [13,19]) the case when each X_i is a 2-element set.

In the situation described in Example 3, the members of G are called “*cubes*” by Störmer [21], and their characteristic functions are called “*cube indicators*”. In fact, in Example 3 we could have taken the set of cube indicators as G, and the set of characteristic functions of all subsets of $X_1 \times \cdots \times X_n$ as L (called “*binary functions*” in [21]).

Example 4. Let L be the set of all *independence systems* on some finite set S, i.e. set systems $I \subseteq \mathscr{P}(S)$ such that $\emptyset \in I$, and $A \in I$, $B \subseteq A$ imply $B \in I$. As each member of L is a set of sets, L is ordered by set inclusion, and it is a lattice under this ordering (in fact a sublattice of $\mathscr{P}\mathscr{P}(S)$). The set G of matroids on S is clearly a subset of L.

Benzaken and Hammer [4] have pointed out that G is a join generating set in L. It is an elementary exercise in matroid theory to show that the G-prime elements of G are precisely those matroids that have only one basis and that condition (i) of Proposition 1 holds. (Observe first that if a matroid I has a unique basis B, then $I=\mathscr{P}(B)$, and if for $m\geqslant 2$ matroids $I_1,\dots,I_m$ we have

$$\mathscr{P}(B)\subseteq I_1\cup\cdots\cup I_m$$

then $B\in I_j$ for some $1\leqslant j\leqslant m$ and thus $\mathscr{P}(B)\subseteq I_j$. Conversely, for any matroid I whose bases are $B_1,\dots,B_m$, $m\geqslant 2$, each $\mathscr{P}(B_i)$ is a matroid, $I\nsubseteq\mathscr{P}(B_i)$ and

$$I=\mathscr{P}(B_1)\cup\cdots\cup\mathscr{P}(B_m).$$

Further, considering any element x of S that is in some but not all bases, we may assume, without loss of generality, that $B_1,\dots,B_k$ contain x and $B_{k+1},\dots,B_m$ do not. Then both independence systems

$$I_1=\mathscr{P}(B_1)\cup\cdots\cup\mathscr{P}(B_k),$$

$$I_2=\mathscr{P}(B_{k+1})\cup\cdots\cup\mathscr{P}(B_m),$$

are matroids distinct from I and $I=I_1\cup I_2$.)

Example 5. Consider any finite loopless graph and let L be the set of its subgraphs. Ordered by the condition that $F\leqslant H$ if and only if F is a subgraph of H,L is a lattice. Let G be the set of those members of L that are complete bipartite graphs. Then G is a join generating set in L, and the G-prime elements of G are those that have only one edge. Obviously condition (i) of Proposition 1 is satisfied. In this context the adjunction of consensus operation used in the formation of consensus sequences has been introduced and studied by Dulmage and Mendelsohn [10], although not for the particular purpose of obtaining the maximal bipartite implicants. An efficient algorithm is presented in [1] for finding all the maximal complete bipartite subgraphs of a graph.

Example 6. Let L_∞ be the set of all n-adic pseudo-Boolean functions, i.e. real-valued functions on $\{0,1\}^n$. Ordered by

$$f\leqslant g \quad \text{if and only if } f(\mathbf{v})\leqslant g(\mathbf{v}) \quad \text{for all } \mathbf{v}\in\{0,1\}^n,$$

L_∞ is an infinite lattice. However, if we fix any particular $f\in L_\infty$ and any finite set S containing the range of f, then

$$L=\{g\in L_\infty\colon \text{ range } g\subseteq S\}$$

is a finite sublattice of L_∞. This fact is in agreement with the view of Davio, Deschamps, Thays [8] and Bioch [5] that pseudo-Boolean functions can be studied as discrete functions. In this framework let $m=\min S$ and let G be the set of those $g\in L$ which are of the form $m+aP$, where a is any non-negative real number, and P is an elementary Boolean conjunction. Then G is a join generating subset of L. The prime implicants of f defined with respect to this set G can be obtained by a pseudo-Boolean analogue of the consensus algorithm ([12]). The validity of this algorithm can be viewed as a consequence of the fact that L and G satisfy condition (i) of Proposition 1. This procedure is remarkably similar to the Blake–Quine consensus method for Boolean functions. Indeed at the adjunction of consensus step the unique prime implicant of $f = m + aP$ and $g = m + bQ$ that is distinct from f and g (if such a prime implicant exists) is

$$m + \min(a, b)P'Q',$$

where $P = P'l$, $Q = Q'\bar{l}$ for some Boolean literal l.

Example 7. Let us suppose that a subset H of a finite lattice L generates L as a meet semilattice, i.e. that every element e of L is the meet of some $m \geqslant 0$ elements $k_1, \ldots, k_m$ of H,

$$e = k_1 \ldots k_m. \tag{6}$$

We shall say that H is a *meet generating* set in L. Expression (6), or its right-hand side, as well as the set $\{k_1, \ldots, k_m\}$, will be referred to as a *conjunctive representation* of e. Any $k \in H$ such that $e \leqslant k$ is called an *implicatum* of e, and the minimal implicata of e are called *prime implicata* (see [8]). This terminology also originates in the theory of Boolean functions, see Example 2 above. For an arbitrary lattice L and meet generating set H, conjunctive representations, implicata and prime implicata are simply disjunctive representations, implicants and prime implicants of the *dual lattice* L^* (obtained from L by the reversal of the partial order relation, after which H becomes a join generating set). If L is the lattice of n-adic Boolean functions, then complementation $f \mapsto 1 - f$ defines a lattice isomorphism between L and L^*. Under this isomorphism, literals correspond to literals, and elementary conjunctions to elementary disjunctions. The consequences of this for the validity of the Boolean resolution method, which is simply the consensus method in conjunctive language, were already discussed in Example 2.

In Section 3 we shall deal with the interplay between implicants and implicata in the context of a particular kind of lattices that includes Boolean function lattices.

3. Disjunctive and conjunctive representations in finite convexities

In this section we shall consider finite convexity spaces and their separation properties as defined e.g. by Van de Vel [23]. A finite *convexity space* is a finite set X, together with a set G of subsets of S called *convex sets*, satisfying the following axioms:

(i) X and $\emptyset$ are convex,
(ii) the intersection of any family of convex sets is convex.

A subset K of X is called *concave* if $X \backslash K$ is convex. A *half-space* is a set that is both convex and concave. The convexity space is said to have property S_1 (*first* *separation property*) if every singleton is convex. It is said to have property S_2 (*second separation property*) if for any two distinct elements x, y of X, there is a half-space H such that $x \in H$, $y \notin H$. Property S_3 (*third separation property*) is said to hold if for every convex set $C \subseteq X$ and every $y \in X \backslash C$, there is a half-space H with $C \subseteq H$, $y \notin H$. Finally, property S_4 (*fourth separation axiom*) is said to hold if for any two disjoint convex sets C, D, there is a half-space H containing C and disjoint from D. It is known that, under the assumption of S_1, S_4 implies S_3 and S_3 implies S_2.

For relevant references on finite convexity, see also Bandelt [2], Duchet [9], Edelman and Jamison [11], Klavžar and Mulder [14], Mulder [17], Mulder and Schrijver [18], Van de Vel [22], and other seminal work surveyed in [23].

Example 8. If F_n is the lattice of n-adic Boolean functions, then associating to each Boolean function f its characteristic set

$$\{\mathbf{v} \in \{0,1\}^n \colon\ f(\mathbf{v}) = 1\}$$

establishes an order-isomorphism between F_n and the inclusion-ordered power set lattice $\mathscr{P}(\{0,1\}^n)$. The characteristic sets of elementary conjunctions, together with the empty set, constitute a convexity on $\{0,1\}^n$, called *graphic cube* convexity (Van de Vel [23]). This provides a geometric language for Boolean functions. Note that literals correspond to half-spaces, and all the four separation properties S_1, S_2, S_3, S_4 hold.

Let $L = \mathscr{P}(X)$ be the inclusion-ordered lattice of all subsets of a set X on which an S_1 convexity has been defined. The set G of convex subsets of X is a join generating set in the distributive lattice $L = \mathscr{P}(X)$. The G-prime elements of G are the empty set and the singletons. A disjunctive representation of any $E \subseteq X$ is a representation of E as a union of convex sets. The implicants of E are the convex sets contained in E and the prime implicants are the maximal convex subsets of E. The following is a straight-forward consequence of Proposition 1.

Proposition 2. *Assume that a convexity on a set X has both the S_1 and S_2 properties. Then the set of all convex sets is a join generating set of $\mathscr{P}(X)$. Every complete consensus sequence for any $E \in \mathscr{P}(X)$ terminates with the family of all prime implicants of E.*

Remark. In the above Proposition the first separation axiom S_1 would not be sufficient. Indeed, consider any finite set $X = \{a, b, c, \ldots\}$ of at least 3 elements, and let the convex sets be defined as $\emptyset$, X, and each one of the singletons. Then $\{a\} \cup \{b\} \cup \{c\} \ldots$ is a disjunctive representation of X, and it constitutes the first and the last representation in a complete consensus sequence for X, even though X is its own unique prime implicant.

Example 9. As a variant of Example 3 consider the following. Let $X = X_1 \times \cdots \times X_n$. Assume that there is a total order on each X_i. Define as convex those sets $Y_1 \times \cdots \times Y_n$ for which each Y_i is an interval in X_i. This convexity does also have the S_1 and S_2 properties, and hence Proposition 2 can be applied. (Remark that this convexity is in fact a *product of total order interval convexities*, while the structure discussed in Example 3 is a *product of free convexities*, see Van de Vel [23].)

The convex sets in this example can be visualized as "boxes" in n-dimensional Euclidean space, and, as in Example 3, those prime implicants of the union of two convex sets $C_1 = Y_1 \times \cdots \times Y_n$ and $C_2 = Z_1 \times \cdots \times Z_n$ which are distinct from C_1 and C_2 are still of the form (5).

Consider a convexity space defined on a set X and let $E \subseteq X$. An *implicatum* of E is defined as any concave subset K of X such that $E \subseteq K$. A *prime implicatum* is a minimal implicatum. If axiom S_1 holds, then clearly every subset of X is the intersection of concave sets, i.e. the intersection of implicata.

In a given convexity space on a set X, consider m half-spaces $h_1, \ldots, h_m$, $m \geqslant 1$. Their intersection

$$h_1 \cap \cdots \cap h_m \tag{7}$$

is a conjunctive representation of a convex set P. Each h_i is an implicatum of P. Clearly the convexity space is S_3 precisely when every convex set can be represented as an intersection (7) of half-spaces. (This is pointed out in [23] as a corollary of a more general result.)

An intersection (7) is said to be an *elementary intersection* if for all i, j

$$h_i \not\subset h_j \quad \text{and} \quad h_i \cap h_j \neq \emptyset.$$

It is easy to verify that a convexity space is S_3 precisely when every non-empty convex set can be represented as an elementary intersection.

Instead of considering the intersection of m half-spaces, we can consider their union

$$h_1 \cup \cdots \cup h_m \tag{8}$$

which gives now a disjunctive representation of a concave set. Each h_i is an implicant of this concave set. Again, it is easy to verify that the convexity space is S_3 precisely when every concave set can be represented as a union (8) of half-spaces.

A union (8) is said to be an *elementary union* if for all i, j

$$h_i \not\subset h_j \quad \text{and} \quad h_i \cup h_j \neq X.$$

It can be seen that a convexity on the set X is S_3 precisely when every concave proper subset of X can be represented as an elementary union.

A convexity is said to have the *Helly property* if, whenever every pair C_i, C_j in a family $C_1, \ldots, C_m$ of convex sets has a non-empty intersection, the intersection $C_1 \cap \cdots \cap C_m$ itself is also non-empty.

Proposition 3. *For any S_3 convexity on a set X the following three conditions are equivalent*:

(i) *the convexity possesses the Helly property*,
(ii) *$\emptyset$ is not an elementary intersection*,
(iii) *X is not an elementary union.*

Proof. Condition (i) obviously implies (ii). Conversely, if (ii) holds, then let us consider for any intersection $C = C_1 \cap \cdots \cap C_n$ of convex sets the intersection of all those half-spaces that contain some C_i. Let us consider now only the minimal members of this family of half-spaces; assuming $C_i \cap C_j \neq \emptyset$ for all i, j, this subfamily of half-spaces defines an elementary intersection. This elementary intersection represents C, and therefore cannot be empty. Thus (ii) implies (i).

The equivalence of (ii) and (iii) is easily seen by taking complements. □

We shall now focus our attention on convexities with the Helly property that possess all the separation properties S_1, S_2, S_3, S_4. Actually, it is enough to assume S_1 and S_2 and the Helly property, as S_3 and S_4 will necessarily follow (see Van de Vel [23,22]). Thus we may speak simply of *separable Helly convexities*.

In any separable Helly convexity on a set X, consider k convex sets $P_1, \ldots, P_k$, and for each P_i let

$$h_{i1} \cap \cdots \cap h_{im(i)}$$

be an elementary intersection representation of P_i. Then, the union

$$(h_{11} \cap \cdots \cap h_{1m(1)}) \cup \cdots \cup (h_{k1} \cap \cdots \cap h_{km(k)}) \tag{9}$$

interpreted as $\emptyset$ for $k = 0$, represents the set $E = P_1 \cup \cdots \cup P_k$; we call (9) a *disjunctive normal form* (DNF) representation of E. Every subset of X has a DNF representation. In the language of Boolean functions this fact is well known and widely used in the graphic cube convexity on $B^n = \{0, 1\}^n$, which is a classical example of a separable Helly convexity.

In any separable Helly convexity on a set X, consider k concave sets $J_1, \ldots, J_k$, and for each J_i let

$$h_{i1} \cup \cdots \cup h_{im(i)}$$

be an elementary union representation of J_i. Then the intersection

$$(h_{11} \cup \cdots \cup h_{1m(1)}) \cap \cdots \cap (h_{k1} \cup \cdots \cup h_{km(k)}) \tag{10}$$

interpreted as X for $k = 0$, represents the set $E = J_1 \cap \cdots \cap J_k$; we call (10) a *conjunctive normal form* (CNF) representation of E. Every subset of X has a CNF representation. In the language of Boolean functions, this fact is well known in $B^n = \{0, 1\}^n$.

In the separable Helly graphic cube convexity of $B^n = \{0, 1\}^n$, if (10) is any CNF representation of a set $E \subseteq B^n$, then the prime implicants of E are precisely the maximal sets of the form

$$h_{1c(1)} \cap h_{2c(2)} \cap \cdots \cap h_{kc(k)}, \tag{11}$$

where for each i, $1 \leqslant c(i) \leqslant m(i)$, and obviously E is the union of all sets of the form (11), due to the rule of distributivity applied to (10). This characterization of prime implicants was established by Kuntzmann in [15] using Boolean consensus. However, Benzaken [3] has proposed a different argument. It consists of the following two facts about B^n (the first of which also follows from Theorem 1.16 of [8]):

(i) if $P_1,\dots,P_{m_1}$ are the prime implicants of a set $E_1\subseteq B^n$ and $Q_1,\dots,Q_{m_2}$ are the prime implicants of $E_2\subseteq B^n$, then the prime implicants of $E_1\cap E_2$ are precisely the maximal sets of the form $P_i\cap Q_j$, $1\leqslant i\leqslant m_1$, $1\leqslant j\leqslant m_2$,

(ii) the half-spaces h_i appearing in the elementary union representation (8) of any concave set $J\subseteq B^n$ are precisely the prime implicants of J.

We shall show that the validity of this distributivity based procedure to obtain the prime implicants is essentially contingent on the Helly property in a convexity space.

Proposition 4. *In a separable Helly convexity, the half-spaces appearing in any elementary intersection* (7) *representing a convex set P are precisely the prime implicata of P. Similarly, the half-spaces appearing in any elementary union representation* (8) *of a concave set J are precisely the prime implicants of J.*

Proof. We give only the proof of the first statement, since the second follows by taking complements in X.

Let (7) be an elementary intersection representing a convex set P. First, we claim that every half-space h containing P contains one of the h_i. If this were not so, then the complementary half-space $\bar{h}=X\setminus h$ would not be disjoint from any h_i. Without loss of generality, assume that

$$\bar{h}\not\subseteq h_i \quad \text{for } i=1,\dots,r,$$

$$\bar{h}\subseteq h_i \quad \text{for } i=r+1,\dots,m.$$

Then $h_1\cap\cdots\cap h_r\cap\bar{h}$ is an elementary intersection and it is, therefore, non-empty. But

$$h_1\cap\cdots\cap h_r\cap\bar{h}=h_1\cap\cdots\cap h_m\cap\bar{h} \tag{12}$$

and thus the right-hand side of (12) is non-empty, contradicting $P\subseteq h$. This proves the claim.

Let J be any implicatum of P. Since P and $X\setminus J$ are disjoint convex sets, S_4 separability implies the existence of a half-space h containing P and disjoint from $X\setminus J$, i.e. contained in J. But we know that h must contain one of the h_i. This h_i is then contained in J. Thus every prime implicatum of P must be one of the h_i, and since we cannot have $h_j\subset h_i$, every h_i is a prime implicatum. □

Proposition 5. *In any separable Helly convexity,*

(D1) *if* (10) *is any CNF representation of a set E, then the prime implicants of E are precisely the maximal sets of the form*

$$h_{1c(1)}\cap\cdots\cap h_{kc(k)}, \tag{13}$$

where for each i, $1\leqslant c(i)\leqslant m(i)$, (D2) *if* (9) *is any DNF representation of a set* E, *then the prime implicata of* E *are precisely the minimal sets of the form*

$$h_{1c(1)} \cup \cdots \cup h_{kc(k)}, \tag{14}$$

where for each i, $1 \leqslant c(i) \leqslant m(i)$.

Proof. Let us first prove (D1), using essentially Benzaken's argument [3]. The two key observations made in the hypercube context remain valid:

(i) If $P_1, \ldots, P_{m_1}$ are the prime implicants of a set E_1 and $Q_1, \ldots, Q_{m_2}$ are the prime implicants of E_2, then the prime implicants of $E_1 \cap E_2$ are the maximal sets of the form $P_i \cap Q_j$, $1 \leqslant i \leqslant m_1, 1 \leqslant j \leqslant m_2$. (This is indeed true in any convexity, in accordance with Theorem 1.16 of [8].)

(ii) The half-spaces appearing in any elementary union representation (8) of a concave set J are precisely the prime implicants of J. (Second statement of Proposition 4.)

Part (D1) follows as in the hypercube case.

From (D1) we can now derive (D2). Let (9) be a DNF of a set E. Let us denote by $\bar{A}$ the complement of any subset A in the convexity space X, i.e. $\bar{A} = X \setminus A$. Then, by De Morgan's law, $\bar{E}$ is

$$(\bar{h}_{11} \cup \cdots \cup \bar{h}_{1m(1)}) \cap \cdots \cap (\bar{h}_{k1} \cup \cdots \cup \bar{h}_{km(k)}). \tag{15}$$

In fact each $\bar{h}_{ij}$ is the complementary half-space of h_{ij} and (15) must be a CNF of $\bar{E}$. Condition (D1) tells us that the prime implicants of $\bar{E}$ are precisely the maximal sets of the form

$$\bar{h}_{1c(1)} \cap \cdots \cap \bar{h}_{kc(k)}, \tag{16}$$

where for each i, $1 \leqslant c(i) \leqslant m(i)$. But a set P is a prime implicant of $\bar{E}$ if and only if $\bar{P}$ is a prime implicatum of E. Therefore, the prime implicata of E are precisely the minimal sets among the complements of the sets of the form (16), i.e., again by De Morgan's law, the minimal sets of the form

$$h_{1c(1)} \cup \cdots \cup h_{kc(k)}$$

where for each i, $1 \leqslant c(i) \leqslant c(k)$. □

Example 10. Finite products of free convexities and finite products of total order interval convexities have the properties S_1, S_2, S_3, S_4. Finite products of total order interval convexities have the Helly property, and hence Proposition 5 applies to the convexity of Example 9. On the other hand, products of free convexities do not have the Helly property, unless each factor has cardinality at most 2—in which case the factors are in fact total order interval convexities. Proposition 5 applies to products of interval convexities, but generally fails for free convexities and products of free convexities.

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